

Exact solution for a binary system of unequal counter-rotating black holes

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A complete solution describing a binary system constituted by two unequal counter-rotating black holes with a massless strut inbetween is presented. It is expressed in terms of four arbitrary parameters: the half length of the two rods representing the black hole horizons σ_1 and σ_2 , the total mass M , and the relative distance R between the centers of the horizons. The explicit form of this solution in terms of physical parameters, i.e., the Komar masses M_1 and M_2 , the Komar angular momenta per unit mass a_1 and a_2 , having a_1 and a_2 opposite signs, and the coordinate distance R , led us to a 4-parameter subclass of solutions in which a set of five physical parameters satisfy a simple algebraic relation. Moreover, the interaction force, provided by the strut, between the black holes results to be of same form as it is for the static double-Schwarzschild case.

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I. INTRODUCTION

The equilibrium configuration of the famous double-Kerr-NUT solution [1] has been extensively studied in the last three decades. Applying regularity conditions on the symmetry axis, the balance equations were first derived by Kihara and Tomimatsu [2, 3]. Hoenselaers solved them analytically for the case of subextreme sources, i.e., non-degenerate black holes, where its analysis revealed the existence of ring singularities outside the horizons, outside of the symmetry axis [4], due principally to the fact that at least one of the Komar masses is negative [5].

Recently Neugebauer and Hennig [6] have shown the non-existence of regular solutions describing equilibrium configurations between two black holes, by using the analytical solution presented by Manko et al. [7]. Additionally, if the parameters do not satisfy the regularity conditions on the symmetry axis, there arise two kind of singularities on the axis, known after Bonnor [8] as *torsion singularity* and *stress singularity*. The first one generates a region with closed timelike curves due to the presence of NUT sources, which lead to finite and semi-infinite singularities along the axis, breaking the *asymptotic* flatness of the solution [9, 10].

The second one represents a strut, a conical singularity [11], which helps us to understand the interaction force between the two bodies by means of the gravitational attraction and a spin-spin interaction.

In the aforementioned equilibrium problem, in the absence of a strut, one always starts by solving the *balance condition*, then the corresponding algebraic variables are substituted into the *axis condition*. Nevertheless, one might choose the opposite way and first solve the equations for avoiding the NUT sources, with the purpose of calculating the massless strut and determine the interaction force between the two black holes. This last approach is more general and complicated to analyze with respect to the equilibrium situation. Nowadays this 5-parameter subclass of the well-known double-Kerr-NUT solution [1] and its analytical representation in terms of physical parameters remain still as an open problem.

One of the first attempts to describe the physical properties of two rotating black holes, was made by Varzugin [12]. He solved the corresponding Riemann-Hilbert problem, in which the *irreducible mass* σ_i is defined as the half length of the rod representing the event horizon of the i -th black hole located on the symmetry axis. First, the axis condition

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for the system of two black holes separated by a massless strut is formulated. The massless strut is introduced in order to avoid the falling of the black holes into each other, due to the gravitational attraction.

Since, by means the Smarr mass formula [13], the irreducible mass is related to the respective surface gravity and area of the black hole horizon, one could express it in terms of physical Komar parameters. Varzugin found an analytical solution describing a binary system constituted by identical counter-rotating black holes, where the interaction force has the same form as the one for the static Schwarzschild case. The corresponding unique σ is described by only three parameters, i.e., the Komar's mass m and angular momentum per unit mass a for the upper black hole [5]. The lower black hole has parameters m and $-a$, and R is the coordinate distance between both constituents.

Later on Manko *et al.* [14] constructed the simplest configuration of two counter-rotating black holes. A 3-parametric solution using the explicit form of σ . They wrote explicit expressions for the Ernst potential on the symmetry axis and for the complete metric outside the symmetry axis. This solution is equatorially antisymmetric [15], where the axis condition is straightforward fulfilled and the total angular momentum of the system vanishes, i.e., $J = 0$.

In the framework of the case of two identical bodies, Bonnor [8, 18] advanced two additional conditions to be satisfied in order to remove the additional contribution provided by the massless spinning rods outside the sources:

$$i) \frac{a_1}{M_1} + \frac{a_2}{M_2} = 0, \quad ii) a_1 + a_2 = 0. \quad (1)$$

The first condition of Eqs.(1) avoids the semi-infinite massless spinning rods, located in the upper and lower parts of the symmetry axis, while the second one, avoids the massless spinning rod of finite length between the two bodies.

On the other hand, it is worthwhile to stress the fact that the most satisfactory solution describing a system of two unequal counter-rotating black holes separated by a massless strut, must be depicted by five physical parameters, i.e., the Komar masses M_1, M_2 of each constituent, their respective Komar angular momenta per unit mass a_1, a_2 (a_1 and a_2 having opposite sign) and the relative coordinate distance R between the centers of the black hole horizons. The main difficulty to accomplish this endeavor is the problem of avoiding the NUT sources in order to be able to provide the explicit form of σ_1 and σ_2 in terms of Komar's physical parameters.

In this work, we first derive a metric describing a system of two rotating black holes by means of the Sibgatullin's method [16, 17]. Then, we apply the axis condition and solve the corresponding equations for the particular case of two unequal counter-rotating black holes separated by a massless strut. We write the solution in terms of σ_1 and σ_2 , as a 4-parameter subclass of the double-Kerr-NUT problem [1]. Later on, we calculate σ_1 and σ_2 by using the Komar integrals for the masses M_i and the angular momenta J_i . We show that the interaction force between the black holes, provided by the strut, has the same form as the one for the static double-Schwarzschild case. Moreover, the 5 parameters satisfy an algebraic relation, which generalizes the two statements made by Bonnor [8, 18] in order to remove additional contributions made by the massless spinning rods outside of the sources, i.e., Eq.(1). In this description, the total angular momentum of the system is *exactly* $J = M_1 a_1 + M_2 a_2$. Notice that it contains only the contributions from the two sources.

The outline of the paper is as follows: In Sec. II all the necessary elements to construct a solution for a binary systems of Kerr sources are presented. In Sec. III a four parametric solution for two counter-rotating black holes separated by a massless strut is analyzed. In Sec. IV the parametrization of the solution in terms of the physical Komar parameters is accomplished. In Sec. V the concluding remarks are given.

II. A SOLUTION FOR A BINARY SYSTEM OF KERR SOURCES

The Papapetrou line element describing stationary axisymmetric spacetimes reads [19]

$$ds^2 = f^{-1} [e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \quad (2)$$

where f, ω, γ are unknown functions depending only on the cylindrical coordinates (ρ, z) . According to the Ernst formalism [20], the vacuum Einstein field equations for these particular stationary axisymmetric spacetimes read:

$$(\text{Re}\mathcal{E})\Delta\mathcal{E} = \nabla\mathcal{E} \cdot \nabla\mathcal{E}, \quad (3)$$

where ∇ and Δ denote the gradient and Laplace operators, respectively, defined in cylindrical coordinates and acting over the complex Ernst potential $\mathcal{E} = f + i\Psi$. For any solution of the equation (3), the metric functions ω and γ of

the line element (2) can be obtained from the following system of differential equations

$$\omega_\rho = -\rho f^{-2} \Psi_z, \quad \omega_z = \rho f^{-2} \Psi_\rho, \quad (4)$$

$$4\gamma_\rho = \rho f^{-2} (|\mathcal{E}_\rho|^2 - |\mathcal{E}_z|^2), \quad 2\gamma_z = \rho f^{-2} \text{Re}(\mathcal{E}_\rho \bar{\mathcal{E}}_z), \quad (5)$$

where the bar over a symbol represents the complex conjugate, $|x|^2 = x\bar{x}$, and the subscript ρ or z denotes partial differentiation. In order to solve the non-linear Eq.(3), we will use the powerful mathematical technique based on the soliton theory known as Sibgatullin's method [16]. The extended double-Kerr-NUT problem [1] can be constructed easily by applying this method as it is done in reference [17] for the case of two bodies, but vanishing electromagnetic field ($\Phi = 0$). Let us start by defining the Ernst potential on the symmetry axis as follows:

$$\mathcal{E}(\rho = 0, z) \equiv e(z) = 1 + \frac{e_1}{z - \beta_1} + \frac{e_2}{z - \beta_2}. \quad (6)$$

The set of complex constants parameters $\{e_j, \beta_j\}$ are a total of eight real parameters related with the multipolar terms. Once we know the value of the Ernst potential on the symmetry axis, the complex potential in the whole space can be obtained from the Sibgatullin's integral

$$\mathcal{E}(\rho, z) = \frac{1}{\pi} \int_{-1}^1 \frac{\mu(\zeta) e(\xi) d\zeta}{\sqrt{1 - \zeta^2}}, \quad (7)$$

whose unknown function $\mu(\zeta)$ satisfies an integral equation

$$\oint_{-1}^1 \frac{\mu(\zeta) [e(\xi) + \tilde{e}(\eta)] d\zeta}{(\xi - \eta) \sqrt{1 - \zeta^2}} = 0, \quad (8)$$

and a normalization condition

$$\frac{1}{\pi} \int_{-1}^1 \frac{\mu(\zeta) d\zeta}{\sqrt{1 - \zeta^2}} = 1, \quad (9)$$

where $\tilde{e}(\eta) \equiv \overline{e(\bar{\eta})}$ and \oint is representing a principal value integral. In addition, $e(\xi)$ is the local holomorphic continuation of $e(z)$ on the complex plane $z + i\rho$, with $\xi = z + i\rho\zeta$, $\eta = z + i\rho\tau$, $\forall \zeta, \tau \in [-1, 1]$. Since $e(z)$ is a rational function, the corresponding $\mu(\zeta)$ can be assumed to be of polynomial form

$$\mu(\zeta) = A_0 + \sum_{n=1}^4 A_n (\xi - \alpha_n)^{-1}, \quad (10)$$

where the coefficients A_0 and A_n are determined by the Eqs.(8)–(9), and the constants α_n represent the location of the sources on the symmetry axis, they are the roots of the following characteristic equation, see Fig.1,

$$e(z) + \tilde{e}(z) = 0. \quad (11)$$

Replacing Eq.(6) into Eq.(11), it is possible to show that the old parameters $\{e_j, \beta_j\}$ and the new ones $\{\alpha_n, \beta_j\}$ are related through the following relations

$$e_1 = \frac{2(\beta_1 - \alpha_1)(\beta_1 - \alpha_2)(\beta_1 - \alpha_3)(\beta_1 - \alpha_4)}{(\beta_1 - \beta_2)(\beta_1 - \bar{\beta}_1)(\beta_1 - \bar{\beta}_2)}, \quad e_2 = \frac{2(\beta_2 - \alpha_1)(\beta_2 - \alpha_2)(\beta_2 - \alpha_3)(\beta_2 - \alpha_4)}{(\beta_2 - \beta_1)(\beta_2 - \bar{\beta}_1)(\beta_2 - \bar{\beta}_2)}. \quad (12)$$

After tedious calculations the solution describing the extended double-Kerr-NUT problem can be straightforward obtained. The Ernst potential \mathcal{E} and the corresponding metric functions f , ω and γ can be written in the following explicit form [21]

$$\mathcal{E} = \frac{E_+}{E_-}, \quad f = \frac{E_+ \bar{E}_- + \bar{E}_+ E_-}{2|E_-|^2}, \quad \omega = -\frac{4\text{Im}[\bar{E}_- G]}{E_+ \bar{E}_- + \bar{E}_+ E_-}, \quad e^{2\gamma} = \frac{E_+ \bar{E}_- + \bar{E}_+ E_-}{2|K_0|^2 \prod_{n=1}^4 r_n}, \quad (13)$$

where

$$\begin{aligned}
 E_{\pm} &= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ \pm 1 & & & & \\ \pm 1 & \mathcal{M} & & & \\ 0 & & & & \\ 0 & & & & \end{vmatrix}, & G &= \begin{vmatrix} 0 & p_1 & p_2 & p_3 & p_4 \\ 1 & & & & \\ 1 & \mathcal{M} & & & \\ 0 & & & & \\ 0 & & & & \end{vmatrix}, & \mathcal{M} &= \begin{pmatrix} \gamma_{11}r_1 & \gamma_{12}r_2 & \gamma_{13}r_3 & \gamma_{14}r_4 \\ \gamma_{21}r_1 & \gamma_{22}r_2 & \gamma_{23}r_3 & \gamma_{24}r_4 \\ \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \end{pmatrix}, \\
 K_0 &= \begin{vmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \end{vmatrix}, & p_n &= z - \alpha_n - r_n, & \gamma_{jn} &= (\alpha_n - \beta_j)^{-1}, & \kappa_{jn} &= (\alpha_n - \bar{\beta}_j)^{-1}, \\
 r_n &= \sqrt{\rho^2 + (z - \alpha_n)^2}.
 \end{aligned} \tag{14}$$

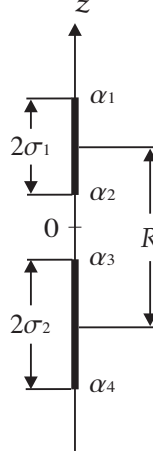


FIG. 1: Location of two unequal Kerr black holes on the symmetry axis represented by the rods of length $2\sigma_1$ and $2\sigma_2$, where the roots satisfy $\sum \alpha_n = 0$ and both bodies are disconnected if the axis condition and $R > \sigma_1 + \sigma_2$ are satisfied.

These last expressions constitute a solution defined by eight parameters represented by the values of the α_n , $n = 1, 2, 3, 4$ and β_j , $j = 1, 2$. However, when α_n are set to be real parameters, the solution describes a solution for a binary system constituted by two Kerr black holes, where the two horizons are defined on the symmetry axis by the intervals $\alpha_1 \geq z \geq \alpha_2$ and $\alpha_3 \geq z \geq \alpha_4$. It is important to notice that the above solution was constructed assuming asymptotic flatness at spatial infinity, where $f \rightarrow 1$, $\gamma \rightarrow 0$ and $\omega \rightarrow 0$ (in the absence of NUT sources), the metric functions γ and ω automatically fulfill the following conditions on the symmetry axis: $\gamma(\alpha_1 < z < \infty) = \gamma(-\infty < z < \alpha_4) = 0$ and $\omega(\alpha_1 < z < \infty) = 0$, thus establishing an *elementary flatness* on the upper part of the symmetry axis.

III. A FOUR-PARAMETRIC SOLUTION

For the case in which the binary system is located on the symmetry axis in such way that the roots α_n satisfy the condition $\sum \alpha_n = 0$, only seven parameters are needed in order to characterize such solution. In order to get rid of the NUT sources between the objects in the lower part of the symmetry axis, i.e., regions $\alpha_3 < z < \alpha_2$ and $-\infty < z < \alpha_4$, thus regularizing the symmetry axis outside the sources, and determining the solution for two counter-rotating black holes with a massless strut inbetween, a well-known conical line singularity [11], we must impose the following two conditions on the metric function ω

$$\omega(\rho = 0, \alpha_2 < z < \alpha_3) = 0, \quad \omega(\rho = 0, -\infty < z < \alpha_4) = 0. \tag{15}$$

We note that the second condition in Eqs.(15) implies the vanishing of the gravitomagnetic monopole (NUT parameter [9]), which also can be determined asymptotically by the Ernst potential on the symmetry axis Eq.(6) as follows

$$\text{Im}[e_1 + e_2] = 0, \tag{16}$$

where e_1 and e_2 are defined in Eq.(12). A straightforward simplification performed with the two conditions in Eq.(15) lead us to the following compact set of algebraic equations

$$\text{Im} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ 1 & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ 0 & \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ 0 & \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \end{bmatrix} = 0, \quad \text{Im} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -\gamma_{11} & -\gamma_{12} & \gamma_{13} & \gamma_{14} \\ 1 & -\gamma_{21} & -\gamma_{22} & \gamma_{23} & \gamma_{24} \\ 0 & \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ 0 & \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24} \end{bmatrix} = 0. \quad (17)$$

These last two equations reduce the seven parametric solution to a five parametric one and the complete metric can be written in a similar form as the one for the double Reissner–Nordström problem [22, 23]. We will restrict our solution to a four parametric subclass. Since the solution Eqs.(13)–(14) involves real constants α_n which determine the location of the two Kerr black hole sources on the symmetry axis, we re-parameterize them as follows

$$\alpha_1 = \frac{R}{2} + \sigma_1, \quad \alpha_2 = \frac{R}{2} - \sigma_1, \quad \alpha_3 = -\frac{R}{2} + \sigma_2, \quad \alpha_4 = -\frac{R}{2} - \sigma_2, \quad (18)$$

where, as mentioned above, σ_1 and σ_2 describe the half length of the two rods representing the black hole horizons and R is the relative distance between the two centers as it is shown in Figure 1. The lengths σ_1 and σ_2 can be written in terms of the Komar physical parameters, i.e., the individual masses M_1 and M_2 , the angular momenta per unit mass a_1 and a_2 , and the coordinate distance R . In our case, the parameters should satisfy an additional relationship imposed by Eq.(17). The Komar integrals for the individual masses M_i and angular momenta J_i can be calculated through the Tomimatsu's formulae [24]:

$$M_i = -\frac{1}{8\pi} \int_{H_i} \omega \Psi_z d\varphi dz, \quad J_i = -\frac{1}{8\pi} \int_{H_i} \omega \left(1 + \frac{1}{2} \omega \Psi_z \right) d\varphi dz. \quad (19)$$

The integrals are over the black hole horizons $H_i = \{\alpha_{2i} \leq z \leq \alpha_{2i-1}, 0 \leq \varphi \leq 2\pi, \rho \rightarrow 0\}$, $i = 1, 2$. Moreover, the total mass M can be considered as the sum of the individual masses M_1 and M_2 once the conditions established in Eq.(15), for regularizing the symmetry axis outside the sources, are fulfilled. In order to solve the system of Eqs.(17), we make a first order expansion in z^{-1} of the Ernst potential on the symmetry axis, Eq.(6), to calculate the total mass M asymptotically, the obtained result reads:

$$\text{Re}[e_1 + e_2] = -2M. \quad (20)$$

Replacing Eq.(12) into Eq.(20) yields the equation

$$\beta_1 + \beta_2 + \bar{\beta}_1 + \bar{\beta}_2 = -2M, \quad (21)$$

implying several possibilities on the relations between the beta-parameters and the total mass M . The simplest choice describing the unequal counter-rotating case is the relation $\beta_1 + \beta_2 = -M$. A simple calculation lead us to the following result

$$\begin{aligned} \beta_{1,2} &= \frac{-M \pm \sqrt{p + iq}}{2}, \\ p &= R^2 - M^2 + 2 \left(\epsilon_1 - \frac{\epsilon_2 R}{M} \right), \\ q &= \frac{2\sqrt{(R^2 - M^2)(M^2 R^2 - \epsilon_2^2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)}}{M(MR + \epsilon_2)}, \\ \epsilon_{1,2} &:= \sigma_1^2 \pm \sigma_2^2, \end{aligned} \quad (22)$$

where the subindexes 1 and 2 are associated with $+$ and $-$ signs, respectively. Therefore, writing the Ernst potential and metric functions in terms of the parameters M , R , σ_1 and σ_2 leads to the following four parametric solution for two unequal counter-rotating black holes:

$$\mathcal{E} = \frac{\Lambda + 2\Gamma}{\Lambda - 2\Gamma}, \quad f = \frac{|\Lambda|^2 - 4|\Gamma|^2}{|\Lambda - 2\Gamma|^2}, \quad \omega = -\frac{2\text{Im}[(\bar{\Lambda} - 2\bar{\Gamma})G]}{|\Lambda|^2 - 4|\Gamma|^2}, \quad e^{2\gamma} = \frac{|\Lambda|^2 - 4|\Gamma|^2}{256\sigma_1^2\sigma_2^2(M^2 R^2 - \epsilon_2^2)^2 r_1 r_2 r_3 r_4}, \quad (23)$$

where

$$\begin{aligned}
\Lambda &= 4\sigma_1\sigma_2 (M^4 - \epsilon_2^2) (r_1 r_2 + r_3 r_4) + [M^4 R^2 + \epsilon_2^2 (R^2 - 2M^2)] (r_1 - r_2) (r_3 - r_4) + (M^4 - 2M^2 R^2 + \epsilon_2^2) \\
&\quad \times [\epsilon_1 (r_1 - r_2) (r_3 - r_4) - 2\sigma_1\sigma_2 (r_1 + r_2) (r_3 + r_4)] - 2i\delta [\sigma_1 (r_1 + r_2) (r_3 - r_4) - \sigma_2 (r_1 - r_2) (r_3 + r_4)], \\
\Gamma &= \sigma_1 (M + \epsilon_2/M) [2\sigma_2 M^2 (\epsilon_2 - R^2) (r_3 + r_4) - R(M^2 - \epsilon_2)^2 (r_3 - r_4)] \\
&\quad - \sigma_2 (M - \epsilon_2/M) [2\sigma_1 M^2 (\epsilon_2 + R^2) (r_1 + r_2) - R(M^2 + \epsilon_2)^2 (r_1 - r_2)] \\
&\quad + i\delta [\sigma_1 (M + \epsilon_2/M) (r_3 - r_4) - \sigma_2 (M - \epsilon_2/M) (r_1 - r_2)], \\
G &= 2z\Gamma + 4\sigma_1\sigma_2 R (M^4 - \epsilon_2^2) (r_1 r_2 - r_3 r_4) + \sigma_1 (R^2 + \epsilon_2) (M^2 - \epsilon_2)^2 (r_1 + r_2) (r_3 - r_4) \\
&\quad + \sigma_2 (R^2 - \epsilon_2) (M^2 + \epsilon_2)^2 (r_1 - r_2) (r_3 + r_4) - 2i\epsilon_2\delta (r_1 - r_2) (r_3 - r_4) - \sigma_1 (M + \epsilon_2/M) \\
&\quad \times \{2\sigma_2 R [\epsilon_2^2 + M^2 (M^2 - R^2 - \epsilon_2)] (r_3 + r_4) + [2M^2 \epsilon_1 (R^2 - \epsilon_2) + (2M^2 - R^2) \epsilon_2^2 - M^4 R^2] (r_3 - r_4) \\
&\quad + i\delta [R (r_3 - r_4) - 2\sigma_2 (r_3 + r_4)]\} + \sigma_2 (M - \epsilon_2/M) \{2\sigma_1 R [\epsilon_2^2 + M^2 (M^2 - R^2 + \epsilon_2)] (r_1 + r_2) \\
&\quad - [2M^2 \epsilon_1 (R^2 + \epsilon_2) + (2M^2 - R^2) \epsilon_2^2 - M^4 R^2] (r_1 - r_2) - i\delta [R (r_1 - r_2) + 2\sigma_1 (r_1 + r_2)]\}, \\
\delta &:= \sqrt{(R^2 - M^2)(M^2 R^2 - \epsilon_2^2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)},
\end{aligned} \tag{24}$$

where r_n can be written in the following parameterized form

$$r_{1,2} = \sqrt{\rho^2 + \left(z - \frac{1}{2}R \mp \sigma_1\right)^2}, \quad r_{3,4} = \sqrt{\rho^2 + \left(z + \frac{1}{2}R \mp \sigma_2\right)^2}, \tag{25}$$

the indices 1,3 and 2,4 run over + and - signs, respectively. Obviously the solution (23)–(24) has not the equatorial antisymmetry property in the sense of [15], the antisymmetry appears only for the case where both constituents are equal.

It is interesting to note that under the transformation $1 \leftrightarrow 2$, $z \rightarrow -z$, which exchange the physical properties and the position of the constituents, will only change the global sign of the metric function ω . The corresponding Ernst potential on the symmetry axis now reads:

$$\begin{aligned}
e(z) &= \frac{e_+}{e_-}, \\
e_{\pm} &= z^2 \mp Mz + \frac{2M^3 - MR^2 - 2M\epsilon_1 \mp 2\epsilon_2 R}{4M} - \frac{i\sqrt{(R^2 - M^2)(M^2 R^2 - \epsilon_2^2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)}}{2M(MR \mp \epsilon_2)}.
\end{aligned} \tag{26}$$

The total angular momentum of the system can be calculated asymptotically by means of a second order expansion in z^{-1} , it is given by

$$J = \frac{\epsilon_2}{2M} \sqrt{\frac{(R^2 - M^2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)}{M^2 R^2 - \epsilon_2^2}}. \tag{27}$$

Under the transformation $1 \leftrightarrow 2$, the total angular momentum changes its sign, i.e., $J = -J_{(1 \leftrightarrow 2)}$. This fact means that Eq.(23) is indeed a solution for the case of two unequal counter-rotating black holes. On the other hand, from the energy-momentum tensor associated with the strut, one obtains the following expression for the interaction force between the black holes [11, 25]

$$\mathcal{F} = \frac{1}{4}(e^{-\gamma_0} - 1) = \frac{M^4 - (\sigma_1^2 - \sigma_2^2)^2}{4M^2(R^2 - M^2)}, \tag{28}$$

where γ_0 is the constant value of the metric function γ evaluated on the corresponding region of the strut. It is worthwhile to mention, that the interaction force between two identical counter-rotating black holes ($M_1 = M_2 = m$, $a_1 = -a_2 = a$) in the non-extreme case: $M = 2m$, $\sigma_1 = \sigma_2 = \sigma$, and in the extreme case: $M = 2m$, $\sigma_1 = \sigma_2 = 0$, is of the same form [12, 14, 26], i.e.,

$$\mathcal{F} = \frac{m^2}{R^2 - 4m^2}. \tag{29}$$

Moreover, in the absence of rotation: $a_1 = a_2 = 0$, $\sigma_1 = M_1$, $\sigma_2 = M_2$, and $M = M_1 + M_2$, we recover the well-known expression for the interaction force between two Schwarzschild black holes [25, 27], i.e.,

$$\mathcal{F} = \frac{M_1 M_2}{R^2 - (M_1 + M_2)^2}. \tag{30}$$

IV. THE PHYSICAL PARAMETRIZATION

The relation between the quantities σ_1 , σ_2 and the physical Komar parameters of the system can be obtained by means of the Tomimatsu's formulae Eqs.(19). Let us use the following simplified form of them [24, 28]:

$$M_i = \frac{\omega_i}{4} [\Psi|_{\rho=0, z=\alpha_{2i}} - \Psi|_{\rho=0, z=\alpha_{2i-1}}], \quad J_i = \frac{\omega_i}{2} (M_i - \sigma_i), \quad i = 1, 2, \quad (31)$$

for the individual Komar masses and angular momenta. ω_i are the constant values of the corresponding metric function ω evaluated over the horizon of each constituent black hole.

The horizons are defined as null hypersurfaces, i.e., $\rho = 0$, $-\sigma_1 \leq z - R/2 \leq \sigma_1$ and $\rho = 0$, $-\sigma_2 \leq z + R/2 \leq \sigma_2$, with a massless strut inbetween. A straightforward calculation leads us to the following system of equations for the individual masses and angular momenta of the black holes:

$$M_1 = \frac{M^2 + \epsilon_2}{2M}, \quad M_2 = \frac{M^2 - \epsilon_2}{2M}, \quad (32)$$

$$J_1 = \frac{M_1}{2M} \sqrt{\frac{(R+M)(MR-\epsilon_2)(M^4-2\epsilon_1 M^2+\epsilon_2^2)}{(R-M)(MR+\epsilon_2)}}, \quad J_2 = -\frac{M_2}{2M} \sqrt{\frac{(R+M)(MR+\epsilon_2)(M^4-2\epsilon_1 M^2+\epsilon_2^2)}{(R-M)(MR-\epsilon_2)}}. \quad (33)$$

From Eq.(32) it is easy to see that the total mass $M = M_1 + M_2$. However, besides this relation one obtains the additional relation

$$\sigma_1^2 - \sigma_2^2 = M_1^2 - M_2^2, \quad (34)$$

replacing Eq.(34) into Eq.(33) leads to the following expressions for σ_i :

$$\sigma_1 = \sqrt{M_1^2 - a_1^2 \frac{(R-M_2)^2 - M_1^2}{(R+M_2)^2 - M_1^2}}, \quad \sigma_2 = \sqrt{M_2^2 - a_2^2 \frac{(R-M_1)^2 - M_2^2}{(R+M_1)^2 - M_2^2}}. \quad (35)$$

Eq.(34) implies the following relation between the five physical parameters:

$$M_1 a_1 + M_2 a_2 + R(a_1 + a_2) - M_1 M_2 \left(\frac{a_1}{M_1} + \frac{a_2}{M_2} \right) = 0. \quad (36)$$

This last relation generalizes the two assumptions Eqs.(1) made by Bonnor [8, 18], in order to remove the contribution arising from the massless spinning rods outside the sources. Notice that Eq.(27) accounts *exactly* for the total angular momentum J as the sum of the individual angular momenta of both constituent black holes, i.e.,

$$J = M_1 a_1 + M_2 a_2. \quad (37)$$

As mentioned above, the condition Eq.(36) means that the contributions to the interaction force made by the two semi-infinite massless spinning rods located on the upper and lower part of the symmetry axis have being removed, as well as the finite massless spinning rod between the two constituents [8, 18]. Hence, Eq.(28) reduces to the simple formula arising for the interaction force between two Schwarzschild black holes.

Moreover, if the condition Eq.(36) between the five independent parameters is not fulfilled, a proper contribution of the spin-spin interaction appears in the expression for the interaction force [28].

Notice that we can recover from Eq.(35) the case of one isolate black hole by imposing the limit $R \rightarrow \infty$ or just setting to zero the physical properties of the other body, i.e., in this case the corresponding mass.

TABLE I: Particular numerical values for the 4-parameter subclass of the Double-Kerr problem.

σ_1	σ_2	M_1	M_2	a_1	a_2	R	J
4.973	1.931	5	2	1.364	-3	8	0.819
1.609	5.881	2	6	5.444	-2.333	10	-3.111
0.681	1.861	1	2	2.5	-1.5	4	-0.5
1.972	1.972	2	2	1	-1	5	0
3	1	3	1	0.667	-2	4	0

Table I shows in the first three rows different sets of numerical values for the masses and for the angular momenta per unit mass of the black holes. The angular momentum of each component having opposite sign. The fourth row displays the case of two equal counter-rotating black holes. The fifth row corresponds to the static case, in which the total angular momentum of the system vanishes, i.e., $J = 0$. In this case, the horizons of the two black holes can reach each other and the system evolves into one Schwarzschild black hole.

Thus, the expressions for \mathcal{E} , f , ω and γ , describing our four parametric solution for two unequal counter-rotating black holes in terms of physical Komar parameters read:

$$\mathcal{E} = \frac{\Lambda + 2\Gamma}{\Lambda - 2\Gamma}, \quad f = \frac{|\Lambda|^2 - 4|\Gamma|^2}{|\Lambda - 2\Gamma|^2}, \quad \omega = -\frac{2\text{Im}[(\bar{\Lambda} - 2\bar{\Gamma})G]}{|\Lambda|^2 - 4|\Gamma|^2}, \quad e^{2\gamma} = \frac{|\Lambda|^2 - 4|\Gamma|^2}{16\sigma_1^2\sigma_2^2[R^2 - (M_1 - M_2)^2]^2 r_1 r_2 r_3 r_4}, \quad (38)$$

where

$$\begin{aligned} \Lambda &= 4\sigma_1\sigma_2 M_1 M_2 (r_1 r_2 + r_3 r_4) - \mu(r_1 - r_2)(r_3 - r_4) \\ &\quad + \sigma_1\sigma_2(R^2 - M_1^2 - M_2^2)(r_1 + r_2)(r_3 + r_4) - i\nu[\sigma_1(r_1 + r_2)(r_3 - r_4) - \sigma_2(r_1 - r_2)(r_3 + r_4)], \\ \Gamma &= -\sigma_1 M_1 [\sigma_2(R^2 - M_1^2 + M_2^2)(r_3 + r_4) + 2M_2^2 R(r_3 - r_4)] \\ &\quad - \sigma_2 M_2 [\sigma_1(R^2 + M_1^2 - M_2^2)(r_1 + r_2) - 2M_1^2 R(r_1 - r_2)] + i\nu[\sigma_1 M_1 (r_3 - r_4) - \sigma_2 M_2 (r_1 - r_2)], \\ G &= 2z\Gamma + 4\sigma_1\sigma_2 M_1 M_2 R (r_1 r_2 - r_3 r_4) + \sigma_1 M_2^2 (R^2 + M_1^2 - M_2^2)(r_1 + r_2)(r_3 - r_4) \\ &\quad + \sigma_2 M_1^2 (R^2 - M_1^2 + M_2^2)(r_1 - r_2)(r_3 + r_4) - i\nu(M_1^2 - M_2^2)(r_1 - r_2)(r_3 - r_4) \\ &\quad - \sigma_1 M_1 \{\sigma_2 R(M_1^2 + 3M_2^2 - R^2)(r_3 + r_4) + 2[\mu + (\sigma_1^2 + \sigma_2^2)M_2^2](r_3 - r_4)\} \\ &\quad + \sigma_2 M_2 \{\sigma_1 R(3M_1^2 + M_2^2 - R^2)(r_1 + r_2) - 2[\mu + (\sigma_1^2 + \sigma_2^2)M_1^2](r_1 - r_2)\} \\ &\quad - i\nu\{\sigma_1 M_1 [R(r_3 - r_4) - 2\sigma_2(r_3 + r_4)] + \sigma_2 M_2 [R(r_1 - r_2) + 2\sigma_1(r_1 + r_2)]\}, \\ \mu &:= (1/2)[(\sigma_1^2 + \sigma_2^2)(R^2 - M_1^2 - M_2^2) - (M_1^2 + M_2^2)R^2 + (M_1^2 - M_2^2)^2], \\ \nu &:= (1/\sqrt{2})(R - M_1 - M_2)\sqrt{a_1^2(R + M_1 - M_2)^2 + a_2^2(R - M_1 + M_2)^2}. \end{aligned} \quad (39)$$

The Ernst potential on the symmetry axis reads:

$$\begin{aligned} e(z) &= \frac{e_+}{e_-}, \\ e_{\pm} &= z^2 \mp Mz - \left(\frac{R}{2} \pm M_1\right)\left(\frac{R}{2} \mp M_2\right) + \left(\frac{R^2 - M^2}{4}\right)F^{\pm 1} \\ &\quad - \frac{R - M}{R + M}\left[\left(\frac{R + M}{2}\right)F^{\pm 1/2} + \frac{i}{\sqrt{2}}\sqrt{a_1^2 F + a_2^2 F^{-1}}\right]^2, \\ F &:= \frac{R + M_1 - M_2}{R - M_1 + M_2}, \end{aligned} \quad (40)$$

σ_i and r_n are given by Eq.(35) and Eq.(25), respectively. Moreover, a_1 and a_2 fulfill Eq.(36).

A. Thermodynamical properties

For each component of the binary system, the Smarr formula for the mass [13] holds, i.e.,

$$M_i = \frac{\kappa_i S_i}{4\pi} + 2\Omega_i a_i M_i = \sigma_i + 2\Omega_i a_i M_i, \quad i = 1, 2, \quad (41)$$

where κ_i is the surface gravity, S_i is the area of the horizon, Ω_i the angular velocity and a_i the angular momentum per unit mass for each constituent black hole. Notice that this last formula implies that $M_i > \sigma_i$. In order to calculate the values of κ_i and Ω_i , one can use the following relations [28, 29]

$$\kappa_i = \sqrt{-\omega_i^{-2} e^{-2\gamma_i}}, \quad \Omega_i = \omega_i^{-1}, \quad (42)$$

being ω_i and γ_i the constant values of the corresponding metric functions ω and γ evaluated over the horizon of each constituent, while $e^{2\gamma}$ is negative at the horizon [28]. By means of the solution Eqs.(38)–(39), it is straightforward to

obtain the following expressions for the angular velocities Ω_i , the surface gravities κ_i , and the area of the horizons S_i

$$\begin{aligned}\Omega_1 &= \frac{a_1[(R - M_2)^2 - M_1^2]}{2M_1(M_1 + \sigma_1)[(R + M_2)^2 - M_1^2]}, \\ \kappa_1 &= \frac{\sigma_1(R + M_1 - M_2)}{2M_1(M_1 + \sigma_1)(R + M)}, \\ S_1 &= \frac{8\pi M_1(M_1 + \sigma_1)(R + M)}{(R + M_1 - M_2)}, \\ \Omega_2 &= \Omega_{1(1 \leftrightarrow 2)}, \quad \kappa_2 = \kappa_{1(1 \leftrightarrow 2)}, \quad S_2 = S_{1(1 \leftrightarrow 2)}.\end{aligned}\tag{43}$$

In the limit when the sources are far away from each other, the angular velocities reduce to:

$$\begin{aligned}\Omega_i &= \frac{a_i F_i}{2M_i(M_i + \sigma_i)}, \\ F_1 &\simeq 1 - \frac{4M_2}{R} + \frac{8M_2^2}{R^2} - \frac{4M_2(M_1^2 + 3M_2^2)}{R^3} + O\left(\frac{1}{R^4}\right), \\ F_2 &\simeq 1 - \frac{4M_1}{R} + \frac{8M_1^2}{R^2} - \frac{4M_1(M_2^2 + 3M_1^2)}{R^3} + O\left(\frac{1}{R^4}\right),\end{aligned}\tag{44}$$

notice that the proper contribution to the angular velocity Ω_i coming from the angular momentum $J_i = M_i a_i$ begins at the third order of the expansion, i.e., $\Omega_i \simeq O(1/R^3)$ [12].

Additionally, in the limit $M_1 = M_2 = m$, $\sigma_1 = \sigma_2 = \sigma$ and $a_1 = -a_2 = a$, our solution reduces to the one for the case of two identical counter-rotating black holes. The unique σ reads

$$\sigma = \sqrt{m^2 - a^2 \left(\frac{R - 2m}{R + 2m} \right)}.\tag{45}$$

Therefore, this particular case belongs to a 3-parameter subclass of the double-Kerr solution [1], where the total angular momentum of the system vanishes, i.e., $J = 0$.

B. Singularities off the axis

Since $M_i > 0$ in (38) there exists no ring singularity off the axis. Nevertheless, if one of the masses is negative, even when the total ADM mass is positive [30], the solution Eqs.(38)–(39) presents such singularities. By setting $f = 0$, this fact can be observed in the following stationary limit surfaces of the Figure 2.

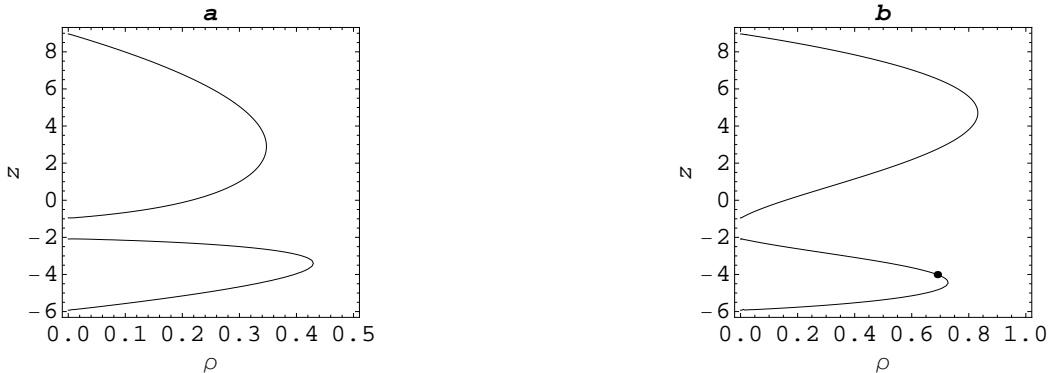


FIG. 2: (a) For positive masses, there exist no singularity off the axis, for the values: $\sigma_1 = 4.973$, $\sigma_2 = 1.931$, $M_1 = 5$, $M_2 = 2$, $a_1 = 1.364$, $a_2 = -3$ and $R = 8$; (b) If one of the masses is negative, there appears the ring singularity off the axis, for the values: $\sigma_1 = 4.973$, $\sigma_2 = 1.931$, $M_1 = 5$, $M_2 = -2$, $a_1 = 0.2$, $a_2 = -3$ and $R = 8$. The ring singularity is located at $\rho \simeq 0.69$, $z \simeq -4.01$.

The localization of such ring singularity can be calculated as one root of the denominator of the Ernst potential Eq.(38).

V. CONCLUDING REMARKS

In this work, we present an exact solution describing a binary system constituted by two unequal counter-rotating black holes with a massless strut inbetween. We derive a 4-parameter subclass involving a simple algebraic relation between the five physical parameters. This relation generalizes, for systems of unequal black holes, the two assumptions made by Bonnor [8, 18] in order to avoid the contribution from the massless spinning rods outside the black holes. Therefore, the interaction force provided by the strut results to be of the Schwarzschild type. This solution reduces to the one for the case when both constituents are identical [14].

On the other hand, in the extreme limit: $\sigma_1 = 0$ and $\sigma_2 = 0$, the unequal and opposite angular momenta per unit mass, in absolute value, are greater than their corresponding positive masses: $|a_i| > M_i > 0$, i.e.,

$$a_1 = \epsilon M_1 \sqrt{\frac{(R + M_2)^2 - M_1^2}{(R - M_2)^2 - M_1^2}}; \quad a_2 = -\epsilon M_2 \sqrt{\frac{(R + M_1)^2 - M_2^2}{(R - M_1)^2 - M_2^2}}; \quad \epsilon = \pm 1. \quad (46)$$

This fact was first pointed out by Herdeiro et al. [31], and it can be obtained also as a consequence of the work of Varzugin [12]. Additionally, in this particular case, the condition established between the five parameters is satisfied only if both constituent black holes are equal, i.e., $M_1 = M_2 = m$ and $a_1 = -a_2 = a$. The total angular momentum of the system vanishes, and the distance R in which the extremality condition occurs [14] reads:

$$R = \frac{2m(a^2 + m^2)}{a^2 - m^2}, \quad |a| > m > 0. \quad (47)$$

It is worthwhile to mention that expressions Eqs.(46), relating the masses and the angular momenta, are of the same form as the relations presented in [32], in the context of a binary system constituted by two extreme Reissner–Nordström black holes with a strut inbetween.

The technical details for removing the NUT sources outside the two rotating black holes is not a trivial problem and it restricts the possibilities for finding exact solutions to more general problems related to the counter/co-rotating cases.

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- [1] D. Kramer and G. Neugebauer, Phys. Lett. **75A**, 259 (1980).
 - [2] M. Kihara and A. Tomimatsu, Prog. Theor. Phys. **67**, 349 (1982).
 - [3] A. Tomimatsu and M. Kihara, Prog. Theor. Phys. **67**, 1406 (1982).
 - [4] C. Hoenselaers, Prog. Theor. Phys. **72**, 761 (1984).
 - [5] A. Komar, Phys. Rev. **113**, 934 (1959).
 - [6] G. Neugebauer and J. Hennig, Gen. Relativ. Gravit. **41**, 2113 (2009).
 - [7] V. S. Manko, E. Ruiz and J. D. Sanabria-Gómez, Class. Quantum Grav. **17**, 3881 (2000).
 - [8] W. B. Bonnor, Class. Quantum Grav. **18**, 1381 (2001).
 - [9] E. Newman, L. Tamburino and T. Unti J. Math. Phys. **4**, 915 (1963).
 - [10] W. B. Bonnor, Gen. Relativ. Gravit. **24**, 551 (1992).
 - [11] W. Israel, Phys. Rev. D **15**, 935 (1977).
 - [12] G. G. Varzugin, Theor. Math. Phys. **116**, 1024 (1998).
 - [13] L. Smarr, Phys. Rev. Lett. **30**, 71 (1973).
 - [14] V. S. Manko, E. D. Rodchenko, E. Ruiz and B. I. Sadovnikov, Phys. Rev. D **78**, 124014 (2008).
 - [15] F. J. Ernst, V. S. Manko and E. Ruiz, Class. Quantum Grav. **23**, 4945 (2006).
 - [16] N. R. Sibgatullin, *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields* (Nauka, Moscow, 1984) (English translation, Springer-Verlag, Berlin, 1991); V. S. Manko and N. R. Sibgatullin, Classical Quantum Gravity **10**, 1383 (1993).
 - [17] E. Ruiz, V. S. Manko and J. Martín, Phys. Rev. D **51**, 4192 (1995).

- [18] W. B. Bonnor and B. R. Steadman, *Class. Quantum Grav.* **21**, 2723 (2004).
- [19] A. Papapetrou, *Ann. Phys. Lpz* **12**, 309 (1953).
- [20] F. J. Ernst, *Phys. Rev.* **167**, 1175 (1968).
- [21] V. S. Manko and E. Ruiz, *Class. Quantum Grav.* **15**, 2007 (1998).
- [22] G. A. Alekseev and V. A. Belinski, *Phys. Rev. D* **76**, 021501(R) (2007).
- [23] V. S. Manko, *Phys. Rev. D* **76**, 124032 (2007).
- [24] A. Tomimatsu, *Prog. Theor. Phys.* **70**, 385 (1983).
- [25] G. Weinstein, *Commun. Pure Appl. Math.* **43**, 903 (1990).
- [26] María E. Gabach Clement, *Class. Quantum Grav.* **29**, 165008 (2012).
- [27] R. Bach and H. Weyl, *Math. Z.* **13**, 134 (1922).
- [28] W. Dietz and C. Hoenselaers, *Ann. Phys.* **165**, 319 (1985).
- [29] A. Tomimatsu, *Prog. Theor. Phys.* **72**, 73 (1984).
- [30] R. Arnowitt, S. Deser, and C. W. Misner, *Phys. Rev.* **122**, 997 (1961).
- [31] C. A. R. Herdeiro and C. Rebelo, *J. High Energy Phys.* 10 (2008)017.
- [32] I. Cabrera-Munguia, V.S. Manko and E. Ruiz, *Gen. Relativ. Gravit.* **43**, 1593 (2011).